8 The Plane Separation Axiom

(8.1) Definition (convex set of points)

Let $\{\mathcal{S}, \mathcal{L}, d\}$ be a metric geometry and let $\mathcal{S}_1 \subseteq \mathcal{S}$. $\mathcal{S}_1$ is said to be convex if for every two points $P, Q \in \mathcal{S}$, the segment $\overline{PQ}$ is a subset of $\mathcal{S}_1$.

1. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are convex subsets of a metric geometry, prove that $\mathcal{S}_1 \cap \mathcal{S}_2$ is convex.

2. If ℓ is a line in a metric geometry, prove that ℓ is convex.

3. Consider the set of ordered pairs (x, y) with $(x-1)^2 + y^2 = 9$, $0 < x < 4$ and $0 < y$. Explain is this set (and when it is) convex.

(8.2) Definition (plane separation axiom (PSA), half planes)

A metric geometry $\{\mathcal{S}, \mathcal{L}, d\}$ satisfies the plane separation axiom (PSA) if for every $\ell \in \mathcal{L}$ there are two subsets H_1 and H_2 of $\mathcal{S}$ (called half planes determined by ℓ) such that

(i) $\mathcal{S} - \ell = H_1 \cup H_2$;

(ii) H_1 and H_2 are disjoint and each is convex;

(iii) If $A \in H_1$ and $B \in H_2$ then $\overline{AB} \cap \ell \neq \emptyset$.

(8.3) Theorem

Let ℓ be a line in a metric geometry. If both H_1, H_2 and H_1', H_2' satisfy the conditions of PSA for the line ℓ then either $H_1 = H_1'$ (and $H_2 = H_2'$) or $H_1 = H_2'$ (and $H_2 = H_1'$).

(8.4) Definition (lie on the same side, lie on opposite sides, side that contains)

Let $\{\mathcal{S}, \mathcal{L}, d\}$ be a metric geometry which satisfies PSA, let $\ell \in \mathcal{L}$, and let H_1 and H_2 be the half planes determined by ℓ. Two points A and B lie on the same side of ℓ if they both belong to H_1 or both belong to H_2. Points A and B lie on opposite sides of ℓ if one belongs to H_1 and one belongs to H_2. If $A \in H_1$, we say that H_1 is the side of ℓ that contains A.

(8.5) Theorem

Let $\{\mathcal{S}, \mathcal{L}, d\}$ be a metric geometry which satisfies PSA. Let A and B be two points of $\mathcal{S}$ not on a given line ℓ. Then

(i) *A and B are on opposite sides of ℓ if and only if $\overline{AB} \cap \ell \neq \emptyset$.*

(ii) *A and B are on the same side of ℓ if and only if either $A = B$ or $\overline{AB} \cap \ell = \emptyset$.*

4. Prove the Theorems 8.3, 8.5 and 8.6.

5. Let ℓ be a line in a metric geometry which satisfies PSA. If P and Q are on opposite sides of ℓ and if Q and R are on opposite sides of ℓ then P and R are on the same side of ℓ.

6. Let ℓ be a line in a metric geometry which satisfies PSA. If P and Q are on opposite sides of ℓ and if Q and R are on the same side of ℓ then P and R are on opposite sides of ℓ.

(8.6) Theorem

Let ℓ be a line in a metric geometry with PSA. Assume that H_1 is a half plane determined by the line ℓ. If H_1 is also a half plane determined by the line ℓ', then $\ell = \ell'$.

(8.7) Definition (edge)

If H_1 is a half plane determined by the line ℓ, then the edge of H_1 is ℓ.

7. Determine are the statements true or false:
(a) If A, B are points, then $\overline{AB}$ is a convex set.
(b) If A, B are points, then $\{A, B\}$ is a convex set.
(c) The intersection of two convex sets is a convex set. (d) The union of two convex sets is a convex set. (e) $\overline{BC} = \overleftrightarrow{BC} \cap \triangle ABC$.

8. Let ℓ be a line in a metric geometry $\{\mathcal{S}, \mathcal{L}, d\}$ which satisfies PSA. We write $P \sim Q$ if P and Q are on the same side of ℓ. Prove that $\sim$ is an equivalence relation on $\mathcal{S} - \ell$. How many equivalence classes are there and what are they?

9. Consider the distance function d_N defined on the Euclidean plane as follows: Let every line other than L_0 have the usual Euclidean ruler, and for the line L_0, let the ruler be $f: L_0 \to \mathbb{R}$ where

$$f((0,y)) = \begin{cases} y, & \text{if } y \text{ is not an integer,} \\ -y, & \text{if } y \text{ is an integer.} \end{cases}$$

(You may assume that this function is a ruler.)

(a) Show that $\{(0,y) \mid \frac{1}{2} \le y \le \frac{3}{2}\}$ is a convex set in $(\mathbb{R}^2, \mathcal{L}_E, d_E)$, the Euclidean plane with the usual Euclidean distance, but not in $(\mathbb{R}^2, \mathcal{L}_E, d_N)$, the Euclidean plane with the new distance.

(b) Find the line segment from $(0,\frac{1}{2})$ to $(0,\frac{3}{2})$ in $(\mathbb{R}^2, \mathcal{L}_E, d_N)$. Show that it is a convex set in $(\mathbb{R}^2, \mathcal{L}_E, d_N)$ but not in $(\mathbb{R}^2, \mathcal{L}_E, d_E)$.

(c) Show that $(\mathbb{R}^2, \mathcal{L}_E, d_N)$, the Euclidean plane with this new distance d_N, does not satisfy PSA, the Plane Separation Axiom.

9 PSA for the Euclidean and Poincaré Planes

Notation **($X^\perp$ or X perp)** If $X = (x,y) \in \mathbb{R}^2$ then $X^\perp$ (read "X perp") is the element $X^\perp = (-y,x) \in \mathbb{R}^2$.

(9.1) Lemma (i) *If $X \in \mathbb{R}^2$ then $\langle X, X^\perp \rangle = 0$.*

(ii) *If $X \in \mathbb{R}^2$ and $X \neq (0,0)$ then $\langle Z, X^\perp \rangle = 0$ implies that $Z = tX$ for some $t \in \mathbb{R}$.*

(9.2) Proposition

If P and Q are distinct points in $\mathbb{R}^2$ then

$$\overleftrightarrow{PQ} = \{A \in \mathbb{R}^2 \mid \langle A-P, (Q-P)^\perp = 0\}.$$

(9.3) Definition (Euclidean half planes)

Let $\ell = \overleftrightarrow{PQ}$ be a Euclidean line. The Euclidean half planes determined by ℓ are

$$H^+ = \{A \in \mathbb{R}^2 \mid \langle A-P, (Q-P)^\perp \rangle > 0\}.$$

$$H^- = \{A \in \mathbb{R}^2 \mid \langle A-P, (Q-P)^\perp \rangle < 0\}.$$

(9.4) Proposition

The Euclidean half planes determined by $\ell = \overleftrightarrow{PQ}$ are convex.

(9.5) Proposition

The Euclidean Plane satisfies PSA.

(9.6) Definition (Poincaré half planes)

If $\ell = {}_aL$ is a type I line in the Poincaré Plane then the Poincaré half planes determined by ℓ are

$$H_+ = \{(x,y) \in \mathbb{H} \mid x > a\}, \qquad H_- = \{(x,y) \in \mathbb{H} \mid x < a\}. \tag{2}$$

If $\ell = {}_cL_r$, is a type II line then the Poincaré half planes determine by ℓ are

$$H_+ = \{(x,y) \in \mathbb{H} \mid (x-c)^2 + y^2 > r^2\}, \qquad H_- = \{(x,y) \in \mathbb{H} \mid (x-c)^2 + y^2 < r^2\}.$$

(9.7) Proposition

The Poincaré Plane satisfies PSA.

1. Prove the Lemma 9.1 and Propositions 9.2, 9.4, 9.5 and 9.7.

2. Prove that the Euclidean half plane H^- is convex.

3. Let ℓ be a line in the Euclidean Plane and suppose that $A \in H^+$ and $B \in H^-$. Show that $\overline{AB} \cap \ell \neq \emptyset$ in the following way. Let

$$g(t) = \langle A + t(B-A) - P, (Q-P)^\perp \rangle \text{ if } t \in \mathbb{R}.$$

Evaluate $g(0)$ and $g(1)$, show that g is continuous, and then prove that $\overline{AB} \cap \ell \neq \emptyset$.

4. If $\ell = {}_aL$ is a type I line in the Poincaré Plane then prove that

a. H_+ and H_- as defined in Equation (2) are convex.

b. If $A \in H_+$ and $B \in H_-$ then $\overline{AB} \cap \ell \neq \emptyset$.

5. For the Taxicab Plane $(\mathbb{R}^2, \mathcal{L}_E, d_T)$ prove that

a. If $A = (x_1,y_1)$, $B = (x_2,y_2)$ and $C = (x_3,y_3)$ are collinear but do not lie on a vertical line then $A-B-C$ if and only if $x_1 * x_2 * x_3$.

b. The Taxicab Plane satisfies PSA.